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Main results of the diploma

Alternating factor groups of Fuchsian triangle groups

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Topics of this presentation

1. Introduction: Why quotients of Fuchsian triangle groups ?

2. Existence of alternating factor groups

3. Search algorithms

4. Conjugacy of normal subgroups in $PSL(2, \mathbb{R})$

1. Introduction: Why quotients of Fuchsian triangle groups ?

Two points of view for **Riemann surfaces**:

- Bernhard Riemann (1851 – dissertation): Riemann surface is the natural maximal domain of some analytic function under analytic continuation.
- Hermann Weyl (1913 – 'Die Idee der Riemannschen Fläche'): A Riemann surface is a one dimensional complex manifold.

Discussion of the different definitions

These two points of view raise in interesting questions:

- If we start with a Riemann surface as an abstract manifold, how do we know that it supports analytic functions ?
- If we develop Riemann surfaces from the point of view of analytic continuation, how do we know that in this way we get all complex manifolds of one complex dimension ?

$\Rightarrow$ Fortunately, it turnes out that these two different views of a Riemann surface are indeed identical.

Uniformization Theorem for Riemann surfaces

Every Riemann surface is the topological quotient with respect to the action of some group Γ of analytic self-maps of one of the three classical geometries of constant curvature:

- Riemann sphere Σ (positive curvature)
- Euclidean (or complex) plane $\mathbb{C}$ (zero curvature)
- Hyperbolic plane $\mathbb{H}$ (negative curvature)

In all cases the analytic self-maps are **Möbius maps**:

$$z \mapsto \frac{az+b}{cz+d}$$

What kind of Riemann surfaces can occur ?

- Quotients of the Riemann sphere: only the sphere itself
- Quotients of the Euclidean plane:
  - Γ trivial $\Rightarrow$ plane itself (once punctured sphere)
  - Γ cyclic $\Rightarrow$ twice punctured sphere
  - Γ generated by two independent translations $\Rightarrow$ torus

Third cast: Hyperbolic plane

So the Uniformization Theorem implies:

- Apart from the non-negative curvature cases, every other Riemann surface is the quotient of the hyperbolic plane $\mathbb{H}$ (upper half-plane in the complex plane) by a group of Möbius (conformal) self-maps of the hyperbolic plane.
- Such groups must be **discrete** (otherwise the quotient structure would not be satisfactory).
- These groups are called **Fuchsian groups**.

Why are triangle groups interesting ?

Can only answered using G. V. Belyi's Theorem (1979), which needs some preparation:

- A function f is called **meromorphic** on a compact Riemann surface $\mathcal{R}$ if it is an analytic map from $\mathcal{R}$ to the Riemann sphere Σ.
- Any such function has a **multiplicity** or **degree** n, in the sense that for any w at the sphere there are exactly n solutions of $f(z) = w, \ z \in \mathcal{R}$. (Where we count multiple solutions in the usual way.)

Belyi's theorem

- For certain values of w the cardinality of the set $f^{-1}(\{w\})$ may be strictly less than n, such w are the **critical values** of the map f: w is a critical value $\Leftrightarrow |f^{-1}(\{w\})| < n$
- **Belyi function**: Meromorphic function f, whose critical values lie in the set $\{0, 1, \infty\}$
- **Belyi's theorem.** A compact Riemann surface $\mathcal{R}$ supports a Belyi function $\Leftrightarrow$ $\mathcal{R}$ is isomorphic to the Riemann surface of some curve $P(z, w) = 0$ whose coefficients are **algebraic** numbers.

Definition of hyperbolic triangle groups

- Let p, q, r integers with $1/p + 1/q + 1/r < 1$.
- Let T denote the hyperbolic triangle with angles π/p, π/q, π/r.
- Let $\Delta^\star$ denote the group generated by the reflections in the sides of T.
- Triangle group $\Delta = \triangle(p, q, r)$ is the subgroup of index 2 in $\Delta^\star$ consisting of conformal transformations.
- This triangle group is determined up to conjugacy in the group of all conformal homeomorphisms of $\mathbb{H}$ by the integers p, q, r.
- $\triangle(p, q, r)$ is is generated by three elliptic generators x, y, z; each with a unique fixed point in $\mathbb{H}$:

$$\triangle(p, q, r) = \langle x, y, z \,|\, x^p = y^q = z^r = xyz = 1\rangle$$

Connection between Fuchsian triangle groups and Riemann surfaces

- Let Γ be a subgroup of finite index in $\Delta = \triangle(p, q, r)$.
- Then Γ is a Fuchsian group and thus $\Gamma\backslash\mathbb{H}$ is a compact Riemann surface.
- The quotient space $\Delta\backslash\mathbb{H}$ is a sphere and the natural projection

$$\tau : \Gamma\backslash\mathbb{H} \mapsto \Delta\backslash\mathbb{H}$$

has at most 3 critical values. (These occur at the projections of the fixed points of the elliptic generators of Δ, but possible if this fixed point is a fixed point of an elliptic generator of Γ, then there will be less than 3 critical values.)

Connection between Fuchsian triangle groups and Riemann surfaces II

- Triangle group $\Delta = \triangle(p, q, r)$
- Finite index subgroup $\Gamma < \Delta$
- Natural projection: $\tau \colon \Gamma\backslash\mathbb{H} \mapsto \Delta\backslash\mathbb{H}$ with at most 3 critical values
- There is a Möbius transformation σ mapping the critical values into $\{0, 1, \infty\}$.
- **Result:** $\beta = \sigma\tau$ is a Belyi function from $\Gamma\backslash\mathbb{H}$ to $\Delta\backslash\mathbb{H}$.

Conclusion of the introduction

Corollary (Belyi, Wolfart, Jones, Singerman). The following statements are equivalent:

1. $\mathcal{R}$ is a Belyi surface (i.e. $\mathcal{R}$ is defined over $\overline{\mathbb{Q}}$).
2. There is a Belyi function $\beta\colon \mathcal{R} \to \Sigma$.
3. $\mathcal{R} \cong \Gamma \backslash U$, where Γ is a finite index subgroup in a cocompact triangle group and U is one of $\mathbb{H}$, $\mathbb{C}$ or Σ.

Background of further investigations

If we look to epimorphisms

$$\varphi\colon \triangle(p,q,r) \to A_n, \qquad \left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1\right)$$

then

$$\Gamma = \mathrm{Kernel}(\varphi) < \triangle(p,q,r)$$

is a subgroup of finite index $|A_n| = \frac{1}{2}\,n!$ and therefore

$$\mathcal{R} = \Gamma\backslash\mathbb{H}$$

is a Riemann surface defined over $\overline{\mathbb{Q}}$ which automorphism group contains A_n.

2. Existence of alternating factor groups

Miller (1901) proved that the classical modular group $PSL(2, \mathbb{Z})$ has among its homomorphic images every alternating group, with the exception of A_3, A_6, A_7, A_8.

Conder (1980) proved that for $n \geq 168$ the alternating group A_n is a Hurwitz group.

In 1981 he proved that

(a) For every $k \geq 7$ all but finitely many alternating groups can be presented as factor groups of $\triangle(2, 3, k)$.

(b) All but finitely many alternating groups can be generated by two elements u, v with $u^2 = v^k = 1$.

More existence theorems

Result of Mustaq/Rota (1992): For nearly all natural numbers n, A_n is a homomorphic image of $\triangle(2, k, l)$ with even $k \geq 6$ and $l \geq 5k - 3$.

Everitt proved in 1994: For all $r \geq 6$, nearly all alternating groups A_n are factor groups of $\triangle(2, 4, r)$.

In 1997 he showed:

(a) For $r \geq 40$ there is a number N so that the group $G = \triangle(3, 5, r)$ has among its homomorphic images the group A_n or S_n for all $n > N$.

(b) For every prime $q \geq 7$ and every $r \geq 4q$, the group $\triangle(3, q, r)$ has the same property.

Final theorem of Everitt

In 2000 Brent Everitt proved the 30 years old conjecture of Higman:

Any Fuchsian group has among its homomorphic images all but finitely many alternating groups.

The proof is constructive and uses coset diagrams. For every Fuchsian group G there is a constant N, so that G surjects the alternating groups A_n for $n \geq N$. N depends only on the signature of the group and can be easily calculated.

Is the theorem of Everitt effective ?

Result of Everitt's proof for some sample groups:

Triangle group	Representation of index n	Lower bound N
$\triangle(2,3,7)$		$N = 168$
$\triangle(2,5,6)$	$n = 105a + 176b + 15$	$N = 18215$
$\triangle(2,5,7)$	$n = 105a + 286b + 15$	$N = 29655$
$\triangle(2,5,9)$	$n = 175a + 48b + 20$	$N = 8198$
$\triangle(2,5,11)$	$n = 66a + 175b + 15$	$N = 11325$
$\triangle(2,5,13)$	$n = 130a + 189b + 26$	$N = 24278$

Result: In general, N is very large.

3. Search Algorithms

Task: Find all epimorphisms for a given triangle group to a given alternating group.

All computations were made using GAP.

Discussion of two methods:

- Built-in algorithm: GQuotients
- Use of low index subgroups algorithm for finitely presented groups

GQuotients

Task:

Determine all epimorphisms up to conjugation of

$$\triangle(p,q,r) = \langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle$$

into a given alternating group A_n.

Commands in GAP:

```
F := FreeGroup("x", "y");
G := F / [F.1^p, F.2^q, (F.1*F.2)^r];
GQuotients(G, AlternatingGroup(n));
```

How does GQuotients work ?

GQuotients is looking for tuples (g_x, g_y) from image group A_n with the following properties:

- g_x and g_y must be images of the FP-generators x and y, e.g. $g_x^p = g_y^q = 1$.
- g_x and g_y must be homomorphic images, e.g. all relations must hold: $(g_x g_y)^r = 1$.
- g_x and g_y must generate A_n.

The method only returns tuples (g_x, g_y) that are unique up to A_n-conjugation.

Low index algorithm for FP groups

Suppose there is epimorphism $\varphi : \Delta \mapsto A_n$. Δ operates on right cosets as A_n. The stabilizer of one point is a subgroup of Δ of index n. The reverse of this statement is also true and can be used to construct the following algorithm:

- Find all subgroups of Δ of index n.
- Test whether the image of the operation of Δ on right cosets is the alternating group.
- Faster test: Size of image must be $|A_n| = \frac{n!}{2}$.

Low index subgroups algorithm in GAP

```
F := FreeGroup("x", "y");
G := F / [F.1^p, F.2^q, (F.1*F.2)^r];
Size_A_n := Factorial(n) / 2;
Subgroups := LowIndexSubgroupsFpGroup(G, n);
All_Images := List(Subgroups,
   sub -> Image(FactorCosetAction(G, sub)));
Interesting_Images := Filtered(All_Images,
   im -> Size(im) = Size_A_n);
Images_Generators := List(Interesting_Images,
   im -> GeneratorsOfGroup(im));
```

How does low index algorithm work ?

With Alexander Hulpke's findings, the algorithm can be described in the following way.

- LowIndex runs over all tuples (g_x, g_y) of permutations out of S_n and tests whether the following conditions hold:
  - $g_x^p = g_y^q = (g_x g_y)^r = 1$
  - $\langle g_x, g_y \rangle$ operates transitively on $m \leq n$ points.
- Result: Subgroups of index n and the operation on the right cosets.
- Last step: Size determination of the images.

Comparison of the two methods

Differences between the algorithms:

- The generation of the tuples is different. (The LowIndex algorithm calculates the permutations pointwise, e.g. first image 1 for all permutations, then image 2 and so on.)
- Conjugacy test is faster in LowIndex (because S_n).

We should use GQuotients if

- the image group is smaller than A_n or S_n, or if
- there are many quotients of small index.

LowIndex should be prefered if

- we only want to determine the existence of epimorphisms.

Genus formula

For a given triangle group $\triangle(p, q, r)$ we look at the existence of epimorphisms into alternating groups A_n. If p, q, r are primes, the *Genus formula* can be used to find some values of n for which no epimorphism into A_n exists.

Theorem (Genus formula):

If the triangle group $\triangle(p, q, r)$ with primes p, q, r has got a subgroup with index n, then we have

$$(p-1)\left[\frac{n}{p}\right] + (q-1)\left[\frac{n}{q}\right] + (r-1)\left[\frac{n}{r}\right] \geq 2n-2,$$

while $[t]$ is the integer part of the rational number t (Gaussian symbol).

Genus formula as transitivity criterion

If the triangle group $G = \triangle(p, q, r)$ with primes p, q, r features an epimorphism $\varphi\colon G \mapsto A_n$ $(n \geq 3)$, then we have

$$(p-1)\left[\frac{n}{p}\right] + (q-1)\left[\frac{n}{q}\right] + (r-1)\left[\frac{n}{r}\right] \geq 2n - 2.$$

This is due to the fact that every epimorphism into A_n corresponds to a subgroup of index n.

The equation above is always true for $n > 3 \max\{p, q, r\}$. In these cases the formula cannot be used to show that there exists no epimorphism into A_n.

Results of the Genus formula

Group	Index n with no epimorphisms into A_n
$\triangle(3,5,7)$	$\{13\}$
$\triangle(5,7,11)$	$\{19\}$
$\triangle(7,11,13)$	$\{19,20\}$
$\triangle(11,13,17)$	$\{21,32\}$
$\triangle(13,17,19)$	$\{25,31,32,33\}$
$\triangle(17,19,23)$	$\{30,31,32,33\}$
$\triangle(19,23,29)$	$\{36,37,45,56\}$
$\triangle(23,29,31)$	$\{42,43,44,45,53,54,55,56,57\}$

4. Conjugacy in $\mathbf{PSL(2,\mathbb{R})}$

Topics of this section:

- If a Fuchsian triangle group Δ contains two normal subgroups N_1 and N_2, how can we determine, if those groups are conjugate to each other in $PSL(2,\mathbb{R})$?
- If they are conjugate, how can we find a suitable subgroup H of $PSL(2,\mathbb{R})$ containing an element h with $N_1^h = N_2$?
- If H can be choosen as a finite-index supergroup of Δ, then the conjugating element h can be easily calculated using integrated systems for computational group theory like GAP.

Conjugacy in $\mathbf{PSL}(2, \mathbb{R})$

Girondo/Wolfart (2005):

> If the $PSL(2, \mathbb{R})$-conjugate surface groups K and K' are both normal subgroups of the triangle group Δ, then $K' = \alpha K \alpha^{-1}$ for some $\alpha \in N(\Delta)$ or $N(\tilde{\Delta})$ where $\tilde{\Delta}$ denotes the normalizer $N(K)$ of K in $PSL(2, \mathbb{R})$.

Using this result, it is possible to prove:

> Let Δ be a triangle group that is contained in only one maximal triangle group $\overline{\Delta}$. If Δ contains two normal subgroups N_1 and N_2 that are conjugate surface groups in $PSL(2, \mathbb{R})$, then there exists an element $h \in \overline{\Delta}$ with $N_1^h = N_2$.

Proof of conjugacy theorem (I)

The proof uses mainly the following statements:

- Previous theorem of Girondo/Wolfart
- The normalizer of a non-cyclic Fuchsian group in $PSL(2, \mathbb{R})$ is again a Fuchsian group.
- Let G be a discrete group of conformal isometries of the hyperbolic plane. If G contains a triangle group as subgroup, then G itself is a triangle group. (Singerman)

Proof of conjugacy theorem (II)

Short version: Surface groups $N_1, N_2 \lhd \Delta$, conjugate in $PSL(2,\mathbb{R})$ $\Rightarrow$ $N_1^h = N_2$ for an element $h \in \overline{\Delta}$.

Proof: Theorem of Girondo/Wolfart states existence of an element h in $N(\Delta)$ or $N(N(N_1))$.

Case 1 – $h \in N(\Delta)$: Since the normalizer is defined as

$$N(\Delta) = N_{PSL(2,\mathbb{R})}(\Delta) = \{\alpha \in PSL(2,\mathbb{R}) \,|\, \Delta^\alpha = \Delta\}$$

we obviously have $\Delta \leq N(\Delta) \leq PSL(2,\mathbb{R})$.

A Fuchsian super-group of a triangle group must be a triangle group:

$$\Delta \leq N(\Delta) \leq \overline{\Delta} < PSL(2,\mathbb{R}) \;\Rightarrow\; h \in \overline{\Delta}.$$

Proof of conjugacy theorem (III)

Case 2 – $h \in N(N(N_1))$:

We have

$$N_1 \trianglelefteq \Delta \;\Rightarrow\; \Delta \leq N(N_1) \leq N(N(N_1)) \leq PSL(2,\mathbb{R}).$$

Since Δ is a triangle group, $N(N_1)$ is also a triangle group and also $N(N(N_1))$ is. Therefore we can conclude:

$$\Delta \leq N(N_1) \leq N(N(N_1)) \leq \overline{\Delta} < PSL(2,\mathbb{R}) \;\Rightarrow\; h \in \overline{\Delta}.$$

Result: In every case, the conjugating element is contained in the maximal triangle group $\overline{\Delta}$.

Numerical example

- Looking for epimorphisms from $\triangle(3,5,5)$ into alternating groups A_n
- Determine whether the kernels are conjugate in $PSL(2,\mathbb{R})$
- Since $\triangle(3,5,5)$ is only contained in the maximal group $\triangle(2,5,6)$, the conjugating element must only be searched in $\triangle(2,5,6)$.

$\Rightarrow$ Embedding of $\triangle(3,5,5)$ into $\triangle(2,5,6)$ and performing conjugacy tests in $\triangle(2,5,6)$

Calculation results

n	$\lvert Epi(\triangle(3,5,5) \mapsto A_n)\rvert$	Conjugate kernels in $\triangle(2,5,6)$
5, 6, 7	2, 2, 3	0, 0, 0
10	22	6 (3 pairs)
11	67	38 (19 pairs)
12	54	40 (20 pairs)
13	24	18 (9 pairs)
15	733	484 (242 pairs)
16	3411	2954 (1477 pairs)
17	3194	2872 (1436 pairs)
18	1564	1374 (687 pairs)
19	377	348 (174 pairs)

Why do only kernel pairs appear?

Lemma: There cannot be a triple (N_1, N_2, N_3) of pairwise $\triangle(2,5,6)$-conjugate kernels that are not conjugate in $\triangle(3,5,5)$.

Proof: Let $\Delta = \triangle(3,5,5)$. Thus $\overline{\Delta} = \triangle(2,5,6)$.

Then $|\triangle(2,5,6) : \triangle(3,5,5)| = 2$ and we have $\overline{\Delta}/\Delta = \{\Delta, x\Delta\}$.

If $N_1 \sim N_2$ then there is an element $\overline{\alpha} \in \overline{\Delta}$ with $N_1 = N_2^{\overline{\alpha}}$. This $\overline{\alpha}$ cannot be an element of Δ, because in this case N_1 and N_2 would be conjugate in Δ. So we have $\overline{\alpha} = x\alpha$ for an element $\alpha \in \Delta$.

If further $N_2 \sim N_3$ there must be an element $\overline{\beta} \in \overline{\Delta}$ with $N_2^{\overline{\beta}} = N_3$. The same argumentation yields $\overline{\beta} = x\beta$ for an element $\beta \in \Delta$.

Together we have $N_3 = N_2^{\overline{\beta}} = N_1^{\overline{\alpha}^{-1}\overline{\beta}} = N_1^{\alpha^{-1}x^{-1}x\beta} = N_1^{\alpha^{-1}\beta}$ and therefore N_1 and N_3 would be conjugate in Δ.

Exceptional cases

There are 7 triangle groups, that are contained in more than one maximal triangle group (Singerman 1972):

- $\triangle(2,7,7) \underset{(9)}{<} \triangle(2,3,7)$, $\triangle(2,7,7) \underset{(2)}{\triangleleft} \triangle(2,4,7)$;
- $\triangle(3,3,7) \underset{(8)}{<} \triangle(2,3,7)$, $\triangle(3,3,7) \underset{(2)}{\triangleleft} \triangle(2,3,14)$;
- $\triangle(3,3,9) \underset{(4)}{<} \triangle(2,3,9)$, $\triangle(3,3,9) \underset{(2)}{\triangleleft} \triangle(2,3,18)$;
- $\triangle(3,8,8) \underset{(10)}{<} \triangle(2,3,8)$, $\triangle(3,8,8) \underset{(2)}{\triangleleft} \triangle(2,6,8)$;
- $\triangle(4,4,5) \underset{(6)}{<} \triangle(2,4,5)$, $\triangle(4,4,5) \underset{(2)}{\triangleleft} \triangle(2,4,10)$;
- $\triangle(7,7,7) \underset{(3)}{\triangleleft} \triangle(3,3,7) \;\Rightarrow$ contained in $\triangle(2,3,7)$, $\triangle(2,3,14)$;
- $\triangle(9,9,9) \underset{(3)}{\triangleleft} \triangle(3,3,9) \;\Rightarrow$ contained in $\triangle(2,3,9)$, $\triangle(2,3,18)$.

All groups on this slide are arithmetic (Takeuchi 1977).

Groups contained in two maximal triangle groups

Can the theorem of Girondo/Wolfart also be reformulated for the seven remaining triangle groups, that are contained in two maximal triangle groups ?

For each group Δ of the seven groups we must answer the following questions:

- What is the normalizer $N(\Delta)$? It will be contained in only **one** maximal group.
- What is $N(N(K))$ if K is a normal subgroup of Δ ? Is this normalizer always contained in the same maximal group for each K ?

Normalizers of the exceptional groups

For five of them the determination is very simple:

$$\triangle(2,7,7) \underset{(2)}{\lhd} \triangle(2,4,7) \Rightarrow N(\triangle(2,7,7)) = \triangle(2,4,7);$$

$$\triangle(3,3,7) \underset{(2)}{\lhd} \triangle(2,3,14) \Rightarrow N(\triangle(3,3,7)) = \triangle(2,3,14);$$

$$\triangle(3,3,9) \underset{(2)}{\lhd} \triangle(2,3,18) \Rightarrow N(\triangle(3,3,9)) = \triangle(2,3,18);$$

$$\triangle(3,8,8) \underset{(2)}{\lhd} \triangle(2,6,8) \Rightarrow N(\triangle(3,8,8)) = \triangle(2,6,8);$$

$$\triangle(4,4,5) \underset{(2)}{\lhd} \triangle(2,4,10) \Rightarrow N(\triangle(4,4,5)) = \triangle(2,4,10);$$

$\triangle(7,7,7)$?

$\triangle(9,9,9)$?

Closer look to $\triangle(\mathbf{7},\mathbf{7},\mathbf{7})$

The inclusion list of Singerman (1972) states:

$$\triangle(7,7,7) \underset{(3)}{\triangleleft} \triangle(3,3,7) \begin{array}{l} \underset{(8)}{<} \triangle(2,3,7) \\ \underset{(2)}{\triangleleft} \triangle(2,3,14) \end{array}$$

To calculate the normalizer of $\triangle(7,7,7)$, this group must be embedded into $\triangle(2,3,7)$ and $\triangle(2,3,14)$.

This can be done using the results of Girondo (2003), who provides subgroup generators for every inclusion between triangle groups.

Embedding of $\triangle(7,7,7)$ into $\triangle(2,3,14)$

The triangle group

$$\triangle(2,3,14) = \langle G, H, I \mid G^2 = H^3 = I^{14} = GHI = 1 \rangle$$

has got the subgroups

$$\langle D, E, F \rangle = \langle GHG, H, H^2GH^2G \rangle \text{ and}$$
$$\langle A, B, C \rangle = \langle H^2GH^2G, GHGH^2GHG, GH^2GH^2 \rangle$$

of index 2 and 6 which are isomorphic to

$$\triangle(3,3,7) = \langle D, E, F \mid D^3 = E^3 = F^7 = DEF = 1 \rangle \text{ and}$$
$$\triangle(7,7,7) = \langle A, B, C \mid A^7 = B^7 = C^7 = ABC = 1 \rangle.$$

$\triangle(7,7,7)$ is normal subgroup of $\triangle(2,3,14)$

Using GAP we get:

```
> F := FreeGroup("x", "y");
> g2314 := F / [F.1^2, F.2^3, (F.1 * F.2)^14];
> G := g2314.1; H := g2314.2;
> g777 := Subgroup(g2314, [H^2*G*H^2*G,
  G*H*G*H^2*G*H*G, G*H^2*G*H^2]);
> Index(g2314, g777);
  6
> IsNormal(g2314, g777);
  true
```

So we have $\triangle(7,7,7) \triangleleft \triangle(2,3,14)$.

Embedding of $\triangle(7,7,7)$ into $\triangle(2,3,7)$

The triangle group

$$\triangle(2,3,7) = \langle G, H, I \mid G^2 = H^3 = I^7 = GHI = 1\rangle$$

has got the subgroups

$$\triangle(3,3,7) = \langle D, E, F \mid D^3 = E^3 = F^7 = DEF = 1\rangle \text{ with index } 8,$$
$$\triangle(7,7,7) = \langle A, B, C \mid A^7 = B^7 = C^7 = ABC = 1\rangle \text{ with index } 24$$

whereby the generators are

$$D = HGHGHGH^2GH^2, E = HGH^2GHGHGH^2, F = H^2G$$

and

$$A = H^2G, B = DH^2GD^2, C = EH^2GE^2.$$

Is $\triangle(7,7,7)$ also normal in $\triangle(2,3,7)$?

Using GAP we get:

- $\triangle(3,3,7) \not\triangleleft \triangle(2,3,7)$ and
- $\triangle(7,7,7) \not\triangleleft \triangle(2,3,7)$.

Since $\triangle(7,7,7) \triangleleft \triangle(2,3,14)$ and $\triangle(2,3,14)$ is maximal, we have $N(\triangle(7,7,7)) = \triangle(2,3,14)$.

Normalizers of the exceptional groups (II)

This is the complete list:

$$\triangle(2,7,7) \underset{(2)}{\lhd} \triangle(2,4,7) \Rightarrow N(\triangle(2,7,7)) = \triangle(2,4,7);$$

$$\triangle(3,3,7) \underset{(2)}{\lhd} \triangle(2,3,14) \Rightarrow N(\triangle(3,3,7)) = \triangle(2,3,14);$$

$$\triangle(3,3,9) \underset{(2)}{\lhd} \triangle(2,3,18) \Rightarrow N(\triangle(3,3,9)) = \triangle(2,3,18);$$

$$\triangle(3,8,8) \underset{(2)}{\lhd} \triangle(2,6,8) \Rightarrow N(\triangle(3,8,8)) = \triangle(2,6,8);$$

$$\triangle(4,4,5) \underset{(2)}{\lhd} \triangle(2,4,10) \Rightarrow N(\triangle(4,4,5)) = \triangle(2,4,10);$$

$$\triangle(7,7,7) \underset{(6)}{\lhd} \triangle(2,3,14) \Rightarrow N(\triangle(7,7,7)) = \triangle(2,3,14);$$

$$\triangle(9,9,9) \underset{(6)}{\lhd} \triangle(2,3,18) \Rightarrow N(\triangle(9,9,9)) = \triangle(2,3,18).$$

Normalizers of subgroups

Small subgroup diagram:

$$\triangle(7,7,7) \underset{(3)}{\triangleleft} \triangle(3,3,7) \begin{array}{l} \underset{(8)}{<} \triangle(2,3,7) \\ \underset{(2)}{\triangleleft} \triangle(2,3,14) \end{array}$$

We already know: $N(\triangle(7,7,7)) = N(\triangle(3,3,7)) = \triangle(2,3,14)$.

Question: Is there a normal subgroup K of $\triangle(3,3,7)$ with $N(N(K)) = \triangle(2,3,7)$?

Equivalent question: Is there a normal subgroup of $\triangle(3,3,7)$ that is also normal in $\triangle(2,3,7)$?

Exploring with GAP

- Define triangle group $G = \triangle(3,3,7)$.
- Use 'LowIndex' to find all subgroups of $\triangle(3,3,7)$ with index 7.
- Label the subgroups with $U_1, \ldots, U_6$. Define with $\varphi_i \colon S_7 \mapsto S_7$ the operation of G on the right cosets G/U_i.

i	1	2	3	4	5	6
Image(φ_i)	A_7	7:3	L(3,2)	A_7	7:3	L(3,2)
|Image(φ_i)|	2520	21	168	2520	21	168
Kernel(φ_i) $\lhd \triangle(3,3,7)$	yes	yes	yes	yes	yes	yes
Kernel(φ_i) $\lhd \triangle(2,3,7)$	no	yes	no	no	no	no

Surprising fact: The kernel of φ_2 is normal in $\triangle(2,3,7)$.

Summary of GAP results

There is a group K with the following properties.

(a) $K \underset{(7)}{\lhd} \triangle(7,7,7) \underset{(3)}{\lhd} \triangle(3,3,7) \underset{(8)}{<} \triangle(2,3,7)$

(b) $K \underset{(21)}{\lhd} \triangle(3,3,7), K \underset{(168)}{\lhd} \triangle(2,3,7)$

(c) Therefore we have $N_{PSL(2,\mathbb{R})}(K) = \triangle(2,3,7)$, although $N_{PSL(2,\mathbb{R})}(\triangle(3,3,7)) = \triangle(2,3,14)$.

(d) K is the kernel of the homomorphism $\varphi_2 \colon \triangle(3,3,7) \mapsto S_7$, $D \mapsto (2\ 4\ 6)(3\ 5\ 7)$, $E \mapsto (1\ 2\ 5)(3\ 6\ 7)$, $F \mapsto (1\ 3\ 5\ 6\ 7\ 4\ 2)$.

(e) Restricted to the subgroup $\triangle(7,7,7)$, the mapping is defined as follows: $A \mapsto (1\ 3\ 5\ 6\ 7\ 4\ 2)$, $B \mapsto (1\ 7\ 3\ 4\ 5\ 2\ 6)$, $C \mapsto (1\ 5\ 7\ 2\ 3\ 6\ 4)$, therefore the image of $\triangle(7,7,7)$ is cyclic.

Conjugacy theorem for exceptional cases

The example has shown, that the theorem of Girondo/Wolfart must be formulated as follows for the seven exceptional groups:

Let Δ be a triangle group that is contained in two different maximal triangle groups $\overline{\Delta}_1$ and $\overline{\Delta}_2$. If Δ contains two normal subgroups N_1 and N_2 that are conjugate surface groups in $PSL(2,\mathbb{R})$, then there exists an element $h \in \overline{\Delta}_1 \cup \overline{\Delta}_2$ with $N_1^h = N_2$.

Numerical example

- Looking for epimorphisms from $\triangle(3,3,7)$ into A_n
- Determine whether the kernels are conjugate in $PSL(2,\mathbb{R})$
- Since $\triangle(3,3,7)$ is contained in the maximal groups $\triangle(2,3,7)$ and $\triangle(2,3,14)$, the following algorithm has to be used twice:
  - Define maximal triangle group $\triangle(2,3,7)$ (resp. $\triangle(2,3,14)$)
  - Define $\triangle(3,3,7)$ as subgroup
  - Determine the epimorphisms from $\triangle(3,3,7)$ into A_n
  - Perform conjugacy tests on the kernels directly in $\triangle(2,3,7)$ (resp. $\triangle(2,3,14)$)

Calculation results

n	$\|\triangle(3,3,7) \mapsto A_n\|$	Conj. in $\triangle(2,3,14)$	Conj. in $\triangle(2,3,7)$
7	2	2 (1 pair)	0
9	5	4 (2 pairs)	0
10	1	0	0
14	128	96 (48 pairs)	?
15	267	220 (110 pairs)	?
16	339	264 (132 pairs)	?
17	110	80 (40 pairs)	?
18	40	20 (10 pairs)	?
19	12	0	?
21	8224	?	?

Summary of the presentation

1. Existence of alternating factor groups:
   Many exists !

2. Search algorithms:
   Algorithm 'LowIndexSubgroups' is faster than 'GQuotients'.

3. Conjugacy of normal subgroups in $PSL(2, \mathbb{R})$:
   Conjugating element always can be found in a maximal triangle group.

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